Linguaggio predicativo e L-strutture

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Istituzioni di logica, 2021-22

Il linguaggio dell'Aritmetica di Peano

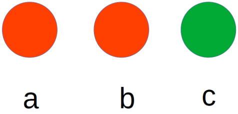
Linguaggio L_A
Costante individuale: 0
Predicati: M^2, U^2
Funtori: s^1, a^2, p^2
Se t, t_1, t_2 sono termini, scriviamo:

- t' al posto di $s^1(t)$;
- $t_1 + t_2$ al posto di $a^2(t_1, t_2)$;
- $t_1 \cdot t_2$ al posto di $p^2(t_1, t_2)$;
- $t_1 > t_2$ al posto di $M^2(t_1, t_2)$;
- $t_1 = t_2$ al posto di $U^2(t_1, t_2)$.

Altri frammenti del linguaggio predicativo

Linguaggio L_C
Costante individuale: p; Predicati: R^1, V^1, S^2; Funtori: n^1

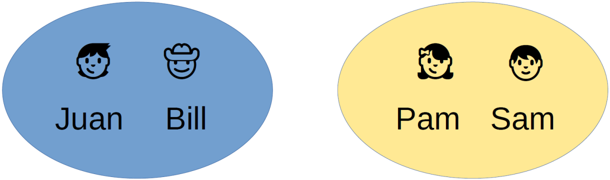
"Coloured dots in a row"



Altri frammenti del linguaggio predicativo

Linguaggio L_D
Costante individuale: p; Predicati: M^1, W^1; Funtori: c^1

"The Dating Game"



La L-struttura dei numeri naturali

$$H_A = \{0, 1, 2, ..\}$$

$$i_A(>) = \left\{ \begin{array}{c} \langle 1, 0 \rangle, \langle 2, 0 \rangle, \\ \langle 2, 1 \rangle, ... \end{array} \right\}$$

$$i_A(=) = \left\{ \begin{array}{c} \langle 0, 0 \rangle, \langle 1, 1 \rangle, \\ \langle 2, 2 \rangle, ... \end{array} \right\}$$

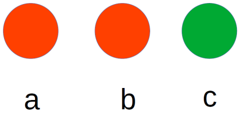
$$i_A(0) = 0$$

$$i_A(') = \{0 \rightsquigarrow 1, ..., 6 \rightsquigarrow 7, ...\}$$

$$i_A(+) = \{\langle 0, 0 \rangle \rightsquigarrow 0, ..., \langle 3, 5 \rangle \rightsquigarrow 8, ...\}$$

$$i_A(\cdot) = \{\langle 0, 0 \rangle \rightsquigarrow 0, ..., \langle 3, 5 \rangle \rightsquigarrow 15, ...\}$$

“Coloured dots in a row”



$$H_C = \{a, b, c\}$$
$$i_C\left(R^1\right) = \{a, b\}$$
$$i_C\left(S^2\right) = \{\langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle\}$$

$$i_C(p) = a$$
$$i_C\left(V^1\right) = \{c\}$$
$$i_C\left(n^1\right) = a \rightsquigarrow b,\ b \rightsquigarrow c,$$
$$c \rightsquigarrow a.$$

Denotazione di termini chiusi

Sia $\langle H, i\rangle = \langle H_C, i_C\rangle$. Vogliamo calcolare $\left|n^1\left(n^1(p)\right)\right|_H^i$.

$$
\begin{aligned}
\left|n^1\left(n^1(p)\right)\right|_H^i &= i\left(n^1\right)\left(\left|n^1(p)\right|_H^i\right) && \text{Def. denotazione}\\
&= i\left(n^1\right)\left(i\left(n^1\right)\left(|p|_H^i\right)\right) && \text{Def. denotazione}\\
&= i\left(n^1\right)\left(i\left(n^1\right)\left(i(p)\right)\right) && \text{Def. denotazione}\\
&= i\left(n^1\right)\left(i\left(n^1\right)(a)\right) && \text{perché } i(p) = a\\
&= i\left(n^1\right)(b) && \text{Def. } i\left(n^1\right)\\
&= c && \text{Def. } i\left(n^1\right)
\end{aligned}
$$

Soddisfacibilità di formule chiuse

Sia $\langle H, i\rangle = \langle H_C, i_C\rangle$. Calcoliamo $\left|R^1(p) \wedge \exists x\left(S^2(p,x)\right)\right|_H^i$.
Abbreviamo tale formula con α.

$$|\alpha|_H^i = 1 \quad \Leftrightarrow \left|R^1(p)\right|_H^i = 1 \text{ e } \left|\exists x\left(S^2(p,x)\right)\right|_H^i = 1$$

$$\Leftrightarrow \begin{cases} |p|_H^i \in i\left(R^1\right) \text{ ed} \\ \text{esiste } k \text{ t.c. } \left|\left(S^2(p,x)\right)_{\underline{k}}^{x}\right|_H^{i_k} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a \in \{a, b\} \text{ ed} \\ \text{esiste } k \text{ t.c. } \left|S^2(p,\underline{k})\right|_H^{i_k} = 1 \end{cases}$$

$$\Leftrightarrow \text{esiste } k \text{ t.c. } \langle i_k(p), i_k(\underline{k})\rangle \in i\left(S^2\right)$$

$$\Leftrightarrow \text{esiste } k \text{ t.c.} \begin{cases} \langle a, k\rangle \in \\ \{\langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle\} \end{cases}$$

Soddisfacibilità di formule chiuse

Sia $\langle H, i \rangle = \langle H_C, i_C \rangle$. Calcoliamo $\left|\forall x \left(S^2\left(n^1(p), x\right) \to V^1(x)\right)\right|_H^i$.
Abbreviamo tale formula con α.

$$|\alpha|_H^i = 1 \quad \Leftrightarrow \text{per ogni } k, \left|\left(S^2\left(n^1(p), x\right) \to V^1(x)\right)_{\underline{k}}^x\right|_H^{i_k} = 1$$

$$\Leftrightarrow \text{per ogni } k, \left|S^2\left(n^1(p), \underline{k}\right) \to V^1\left(\underline{k}\right)\right|_H^{i_k} = 1$$

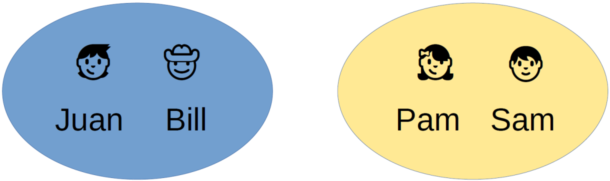
$$\Leftrightarrow \text{per ogni } k, \begin{cases} \left|S^2\left(n^1(p), \underline{k}\right)\right|_H^{i_k} = 0 \\ \text{o } \left|V^1\left(\underline{k}\right)\right|_H^{i_k} = 1 \end{cases}$$

$$\Leftrightarrow \text{per ogni } k, \begin{cases} \left\langle \left|n^1(p)\right|_H^{i_k}, \ \left|\underline{k}\right|_H^{i_k} \right\rangle \notin I\left(S^2\right) \\ \text{o } \left|\underline{k}\right|_H^{i_k} \in I\left(V^1\right) \end{cases}$$

1 caso: $k \in \{a, b\} \quad \Rightarrow \quad \langle b, k \rangle \notin \{\langle a, b \rangle, \langle a, c \rangle, \langle b, c \rangle\}$: ok!!!

2 caso: $k = c \quad \Rightarrow k \in \{c\}$: ok!!!

“The Dating Game”



$$H_D = \{\text{Juan, Bill, Pam, Sam}\}$$
$$i_D\left(M^1\right) = \{\text{Juan, Bill, Sam}\}$$
$$i_D\left(c^1\right) = \text{Juan} \rightsquigarrow \text{Bill, Bill} \rightsquigarrow \text{Juan}$$
$$\text{Pam} \rightsquigarrow \text{Sam, Sam} \rightsquigarrow \text{Pam}$$

$$i_D\left(p\right) = \text{Pam}$$
$$i_D\left(W^1\right) = \{\text{Pam}\}$$